A STABILITY AND SENSITIVITY ANALYSIS OF PARAMETRIC FUNCTIONS IN A SEDIMENTATION MODEL

UN ANALISIS DE ESTABILIDAD Y SENSIBILIDAD DE LAS FUNCIONES DEFINIDAS POR PARÁMETROS EN UN MODELO DE SEDIMENTACIÓN

CARLOS D. ACOSTA
PhD, Departamento de Matemáticas y Estadística, Universidad Nacional de Colombia, Manizales, cdacostam@unal.edu.co

RAIMUND BÜRGER
Dr. rer. nat., CFMA y Departamento de Ingeniería Matemática, U. de Concepción, Chile, rburger@ing-mat.udec.cl

CARLOS E. MEJIA
PhD, Escuela de Matemáticas, Universidad Nacional de Colombia, Medellín, cemejia@unal.edu.co

Received for review December 13th, 2012, accepted October 18th, 2013, final version December, 08th, 2013

ABSTRACT: This paper deals with the reliable and efficient numerical identification of parameters defining the flux function and the diffusion coefficient of a strongly degenerate parabolic partial differential equation (PDE), which is the basis of a mathematical model for sedimentation-consolidation processes. A zero-flux initial-boundary value problem (IBVP) posed for this PDE describes the settling of a suspension in a column. The parameters for a given material are estimated by the repeated numerical solutions of the IBVP (direct problem) under systematic variation of the model parameters, with the aim of successively minimizing a cost functional that measures the distance between a space-dependent observation and the corresponding numerical solution. Two important features of this paper are the following. In the first place, the method proposed for the efficient and accurate numerical solution of the direct problem. We implement a well-known explicit, monotone three-point finite difference scheme enhanced by discrete mollification. The mollified scheme occupies a larger stencil but converges under a less restrictive CFL condition, which allows the use of a larger time step. The second feature is the thorough sensitivity and stability analysis of the parametric model functions that play the roles of initial guess and observation data, respectively.

Keywords: Sedimentation of suspensions, sensitivity analysis, degenerate parabolic equation, parameter estimation, discrete mollification.

RESUMEN. Este artículo se dedica a la identificación numérica confiable y eficiente de los parámetros que definen la función de flujo y el coeficiente de difusión en una ecuación diferencial parcial de tipo parabólico fuertemente degenerada que es la base de un modelo matemático para procesos de sedimentación-consolidación. Para esta ecuación, el problema de valor inicial con valores en la frontera (IBVP) en el que el flujo es nulo, describe el asentamiento de una suspensión en una columna. Los parámetros para un material dado se estiman con base en repetidas soluciones numéricas del problema directo (IBVP) con una variación sistemática de los parámetros del modelo, con el objeto de minimizar sucesivamente un funcional de costo que mide la distancia entre una observación dependiente de tiempo y la correspondiente solución numérica. En este artículo se destacan dos aspectos. El primer aspecto es que en el método propuesto para la solución numérica eficiente y acertada del problema directo, se implementa un esquema explícito monótono bien conocido basado en diferencias finitas que usan tres puntos mejorado por molificación discreta. El esquema molificado utiliza una malla de más puntos pero converge con una condición CFL menos restrictiva, lo cual permite usar pasos temporales más grandes. El segundo aspecto es el exhaustivo análisis de sensibilidad y estabilidad de las funciones definidas por parámetros en el modelo y que juegan los papeles de aproximación inicial y dato observado, respectivamente.

Palabras claves: Sedimentación de suspensiones, análisis de sensibilidad, ecuación parabólica degenerada, estimación de parámetros, molificación discreta.

1. INTRODUCTION

\[ u_t + f(u)_x = A(u)_{xx}, \]

\[(x, t) \in \Omega_T := (0, L) \times (0, T], L > 0, T > 0 \quad (1a) \]

\[ u(x, 0) = u_0(x), \quad x \in [0, L], \quad (1b) \]

\[ f(u) - A(u)_x|_{x=0} = \psi_0(t), \quad t \in (0, T], \quad (1c) \]

\[ f(u) - A(u)_x|_{x=L} = \psi_L(t), \quad t \in (0, T], \quad (1d) \]
where $A$ is an integrated diffusion coefficient, i.e.,

$$A(u) = \int_0^u a(s) \, ds, \quad a(u) \geq 0. \quad (2)$$

The diffusion function $a$ is assumed to be integrable and is allowed to vanish on $u$-intervals of positive length, on which (1a) turns into a first-order hyperbolic conservation law, so that (1a) is a strongly degenerate parabolic. On the other hand, we assume that $f$ is piecewise smooth and Lipschitz continuous. Under suitable choices of $u_0, f, a, \psi_0$ and $\psi_L$ the IBVP (1) may describe a variety of real-world applications like traffic flow [9]. We focus our attention on Equation (1) as a model of the sedimentation-consolidation process of a solid-liquid suspension [8].

It is well known that solutions of (1a) are, in general, discontinuous even if $u_0$ is smooth, and need to be defined as weak solutions along with an entropy condition to select the physically relevant solution, the entropy solution. For the definition, existence and uniqueness of entropy solutions of (1) we refer to [7, 8, 10].

In the present work we are interested in a stability and sensitivity analysis of the parametric model functions. In order to perform the tests, we first proceed with a numerical estimation procedure based on repeated numerical solutions of the direct problem (1) under successive variation of parameters appearing in the coefficient functions $f$ and $a$. In this phase the main components are the efficient and stable solver of the direct problem and the optimization procedure based on the Nelder-Mead Simplex Method. Our goal is the stability and sensitivity analysis of the resulting inverse problem. Theoretical aspects related to identifiability are not our concern in this paper (but cf., e.g., [11]). By sensitivity analysis we mean an intensive set of tests for the numerical identification of parameters with or without noisy observation data. Our approach follows the methodology of [4] but we acknowledge the existence of other ways to perform a sensitivity analysis, for instance [19].

1.2. Related work and outline of the paper

The discrete mollification method is a convolution-based filtering procedure suitable for the regularization of ill-posed problems and for the stabilization of explicit schemes for the solution of PDEs. For the numerical identification of diffusion coefficients by discrete mollification, see [16] and its references.

Inverse problems for strongly degenerate parabolic equations are of particular interest in the context of the sedimentation-consolidation model. In fact, in applications such as wastewater treatment and mineral processing, the reliable extraction of material-specific parameters appearing in the model functions $f$ and $a$ from laboratory experiments allows the operation and control of continuous clarifier-thickeners handling the same material to be simulated [10, 21]. For the special case $A \equiv 0$, i.e., when effects of sediment compressibility are absent or negligible, (1a) reduces to a first-order nonlinear conservation law, portions of the function $f$ can be identified by comparing observed space-time trajectories of concentration discontinuities, with trajectories appearing in closed-form solutions for piecewise constant initial concentrations [6, 14]. In the presence of sediment compressibility, closed-form solutions are not available and one has to resort to numerical techniques to solve the parameter identification problem [5, 11].

The paper is organized as follows. Section 2 presents the sedimentation-consolidation model along with details on the schemes for the solution of the direct problem, including a brief description of the mollification method. Section 3 deals with the parameter identification problem, the proposed algorithm, the sensitivity analysis and the effect of noisy observation data. This section ends with some conclusions.

2. THE APPLICATION OF THE MATHEMATICAL MODEL

2.1. Sedimentation model

According to [8, 10] and the references cited in these works, (1) can be understood as a model for the settling of a flocculated suspension of small solid particles dispersed in a viscous fluid, where $u = u(x, t)$ is the local solid concentration as a function of height $x$ and time $t$. For batch settling in a closed column of height $L$ we set $\psi_0 = 0$ and $\psi_L = 0$; the function $u_0$ denotes the initial solid concentration. The material specific function $f$ describes the effect of hindered settling.
We employ here the following typical parametric expression:

\[
 f(u) = \begin{cases} 
 v_\infty u \left( 1 - \frac{u}{u_{\text{max}}} \right)^c & 0 \leq u \leq u_{\text{max}}, \\
 0 & \text{otherwise},
\end{cases}
\]

where \( v_\infty < 0 \) is the settling velocity of a single particle in an unbounded fluid, \( C > 1 \) is a dimensionless exponent that quantifies how rapidly the settling velocity decreases (as an absolute value) with increasing solids concentration, and \( 0 < u_{\text{max}} \leq 1 \) is a (nominal) maximal solids concentration. The function \( A \) is given by (2), where we define

\[
 a(u) = -\frac{f(u)\sigma_e(u)}{(\rho_s-\rho_l)gu},
\]

where \( \rho_s \) and \( \rho_l \) are the solid and fluid densities, respectively, \( g \) is the acceleration of gravity and \( \sigma'_e(u) = \frac{d\sigma_e}{du} \) is the derivative of the material specific solid stress function \( \sigma_e \).

Among several proposed semi-empirical approaches for \( \sigma_e \), we chose the power law type function

\[
 \sigma_e(u) = \begin{cases} 
 0 & \text{for } 0 \leq u \leq u_c, \\
 \sigma_0 \left( \frac{u}{u_c} \right)^\beta - 1 & u_c < u,
\end{cases}
\]

with material-dependent parameters \( \sigma_0 > 0 \) and \( \beta > 1 \). The values of \( \beta \), \( \sigma_0 \) and \( u_c \) characterize the compressibility of the sediment formed by a given material.

Values of the primitive \( A(u) \) usually have to be determined by numerical quadrature. However, if \( f \) and \( a \) are given by (3)–(4) and \( \beta \) is an integer, then \( A(u) \) can be evaluated in closed form by \( A(u) = 0 \) for \( 0 \leq u \leq u_c \) (equation (1a) is strongly degenerate) and \( A(u) = A(u) - A(u_c) \) for \( u > u_c \), where the function \( A \) is defined by

\[
 A(u) := \frac{v_\infty \sigma_0}{\Delta_p} \frac{gu_c^\beta u_{\text{max}}^c}{\Delta t} \times \sum_{k=1}^{\beta} \left( \prod_{l=1}^{k-\beta+1-i} \frac{C+l}{C+l} \right) u_{\text{max}}^c - u)^{C+k} u^{\beta-k}.
\]

2.2. Discrete mollification

The discrete mollification method \([17, 18]\) consists in replacing a set of data \( y = \{y_j\}_{j \in \mathbb{Z}} \) by its mollified version \( J_\eta y \), where \( J_\eta \) is the discret mollification operator defined by

\[
 [J_\eta y]_j := \sum_{i=-\infty}^{\eta} \omega_i y_{j-i}, \quad j \in \mathbb{Z}.
\]

The support parameter \( \eta \in \mathbb{N} \) indicates the width of the mollification stencil, and the weights \( \omega_i \) satisfy \( \omega_i = \omega_{-i} \) and \( 0 \leq \omega_i \leq \omega_{i-1} \) for \( i = 1, \ldots, \eta \) along with \( \omega_{\eta} + \cdots + \omega_{\eta-1} + \omega_{\eta} = 1 \). The weights \( \omega_i \) are obtained by numerical integration of a suitable truncated Gaussian kernel. Details can be found in \([1, 2, 3, 16, 20]\).

2.3. Discretization of the direct problem

The domain \( \Omega_T \) is discretized by a standard Cartesian grid by setting \( x_j := j\Delta x, j = 0, \ldots, N \), where \( N\Delta x = L \) and \( t_n := n\Delta t, n = 0, \ldots, M \), where \( M \Delta t = T \).

We denote by \( u^n_j \) an approximate value of the cell average of \( u = u(x, t) \) over the cell \([x_j, x_{j+1}]\) at time \( t = t_n \) and correspondingly set

\[
 u^0_j = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0(x) \, dx, \quad j = 0, \ldots, N - 1.
\]

We solve (1) numerically using two convergent finite difference methods. The first one \([8, 13]\) has the following form, where \( \lambda := \Delta t/\Delta x \) and \( \mu := \Delta t/\Delta x^2 \):

\[
 u^{n+1}_j = u^n_j - \lambda \Delta_x F^{EO}(u^n_{j-1}, u^n_j) + \mu \left( A(u^n_{j+1}) - A(u^n_j) + A(u^n_{j-1}) \right).
\]

Here \( F^{EO} \) stands for the well-known Engquist-Osher numerical flux \([12]\), and \( \Delta_x \) denotes the standard forward difference operator. Scheme (5) is monotone and convergent under the CFL condition

\[
 \lambda \|f'\|_{\infty} + 2\mu \|a\|_{\infty} \leq 1.
\]

The second finite difference method is the mollified scheme \([2]\), which is also monotone and convergent and takes the form

\[
 u^{n+1}_j = u^n_j - \lambda \Delta_x F^{EO}(u^n_{j-1}, u^n_j) + 2\mu \eta \left( [J_\eta A(u^n_j)]_j - A(u^n_j) \right),
\]

where \( J_\eta \) is the mollification operator defined by

\[
 [J_\eta y]_j := \sum_{i=-\infty}^{\eta} \omega_i y_{j-i}, \quad j \in \mathbb{Z}.
\]
with
\[ C_\eta := \left[ \sum_{j=-\eta}^{\eta} J^2 \omega_\eta \right]^{-1}. \]

This is an explicit method and has the convenient CFL condition
\[ \lambda \| J' \|_\infty + 2 \mu \varepsilon_\eta \| a \|_\infty \leq 1, \quad \text{(8)} \]

where \( \varepsilon_\eta < 1 \). (For the particular mollification weights considered herein, we obtain \( \varepsilon_3 = 0.7130, \varepsilon_5 = 0.3969 \) and \( \varepsilon_9 = 0.1960 \).) Clearly, condition (8) is more favorable than (6) since it shows that for a given value of \( \Delta t \), mollified schemes may proceed by larger time steps. See [2] for more details on this scheme.

3. SENSITIVITY AND STABILITY ANALYSIS

3.1. Parameter identification problem

The inverse problem can be formulated as follows: given observation data \( u^{obs}(x) \) at the final time \( T > 0 \) and functions \( u_0, \psi_0 \) and \( \psi_L \), find the flux \( f \) and the diffusion function \( a \) such that the entropy solution \( u(x, T) \) of problem (1) is as close as possible to \( u^{obs}(x) \) in some suitable norm. The inverse problem is solved by minimizing the cost function
\[
J(u(\cdot, T)) = \frac{1}{2} \int_0^T \left[ u(x, T) - u^{obs}(x) \right]^2 dx.
\]

Since the functions \( f \) and \( a \) depend on a vector of parameters, the inverse problem corresponds to the following parameter identification problem:

**Minimize** \( J(p) \) \( \text{w. r. t. parameter vector } p \). \( \text{(PI)} \)

The functions \( f \) and \( a \) are associated to the current parameter vector \( p \).

We define the piecewise constant function \( u^\Delta \) by \( u^\Delta(x, t) = u^\Delta_j \) for \( x \in [x_j, x_{j+1}) \) and \( t \in [t_n, t_{n+1}) \) for \( j = 0, \ldots, N - 1 \) and \( n = 0, \ldots, M - 1 \) and replace \( u^{obs} \) by a piecewise constant function \( u^{obs, \Delta} \) formed by cell averages as follows:

\[
u^{obs, \Delta}(x) = u_j^{obs} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u^{obs}(x) \, dx \quad \text{for } x \in [x_j, x_{j+1}),\]

where \( j = 0, \ldots, N - 1 \). The parameter dependent cost function is

\[ f_j^A(p) = \frac{1}{2} \int_0^T \left[ u^\Delta(x, T) - u^{obs, \Delta}(x) \right]^2 dx = \frac{\Delta x}{2} \sum_{j=0}^{N-1} \left[ u^\Delta_j - u_j^{obs} \right]^2. \]

This yields a discrete version of (PI) given by

\[
\begin{cases}
\text{minimize } f_j^A(p) \text{ w. r. t. } p \text{ a and associated to current } p.
\end{cases} \quad \text{(PIj)}
\]

There are many options for the numerical implementation of the optimization procedure. We selected a globalized bounded Nelder-Mead Method with restarts (MATLAB function \( \text{fminsearch} \), see [15] for details), which is a major improvement over the basic simplex method. The strategy is described by the following algorithm. Suppose \( p_j = (p_j^1, \ldots, p_j^K) \), that is, \( K \) different parameters are sought.

**Step 1** Input \( p_0, \epsilon \)

**Step 2** for \( j = 1 \) to \( M \)

\[ p_j = \text{fminsearch} \left( f_j^A, p_{j-1} \right). \]

If \( \max_{1 \leq k \leq K} \left| \frac{p_j^k - p_{j-1}^k}{p_{j-1}^k} \right| \leq \epsilon \) then break, end.
3.2. Numerical examples

The reference solution is generated by the corresponding numerical scheme (5) or (7) on a very fine grid. For examples 1, 2 and 3 we consider batch settling in a column of height $L = 0.16$ m and parameter values $u_{\text{max}} = 0.5$, $g = 9.81 \text{m/s}^2$, $\rho_s - \rho_f = 1660 \text{kg/m}^3$, $v_{\infty} = -2.7 \times 10^{-4} \text{m/s}$, $C = 21.5$, $\beta = 5$, $u_c = 0.07$ and $\sigma_0 = 1.2 \text{Pa}$.

The objective is to obtain an accurate identification of the parameters $u_c$, $\sigma_0$, and $C$ in eight different instances described in Table 1. Our experiments include clean and noisy observation data. Data at the instant $T = 800 \text{s}$ will play the role of $u_{\text{obs}}$. Figures 1 and 2 (a) show the reference solution over the whole computational domain and the profile at $T = 800 \text{s}$, respectively. The restarting parameter and the tolerance parameter for the optimization are $M = 10$ and $\epsilon = 10^{-4}$, respectively.

Example 1: Sensitivity to mollification parameters. Clean observation data (no noise added) and $\Delta x = L/256$. The results are summarized in Table 2. Here, $j$ denotes the number of calls to the fminsearch algorithm, $p_j$ is the vector of parameter values found, $E_j$ is the required number of computed solutions of the direct problem, $e_{\infty}$ is the maximum relative error in the result for each parameter (usually due to $\sigma_0$), and $CPU$ denotes the total CPU time of each run.

Example 2: Sensitivity to initial guess. We randomly generate 100 initial guesses and carry out the identification task. Each initial guess $p_0 = (u_c^0, \sigma_0^0, C^0)^T$ is generated in the form

$$u_c^0 = (1 + 0.3 \xi_1) u_c, \sigma_0^0 = (1 + 0.3 \xi_2) \sigma_0,\ C^0 = (1 + 0.3 \xi_3) C,$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$ is a uniformly distributed random vectorial variable whose components are between $-1$ and $1$. The results are indicated in Table 3. Here, the average $\bar{e}_{\infty}$ of $e_{\infty}$ and its standard deviation $\sigma$ are included. Additionally, column “# restarts” stands for the number of calls of fminsearch and $E_j$ for the number of solutions of the direct problem.

Example 3: Effect of noisy observation data. We randomly generate 100 final profiles and associate them to the previously generated initial guesses. The corrupted profile is generated as follows:

$$u_j^\epsilon = (1 + \epsilon \varphi_j) u_{\text{obs}}(x_j), \quad j = 0, ..., N - 1,$$

where $\epsilon = 0.01, 0.03$ and $0.05$, and $\varphi_j$ is a uniformly distributed random variable assuming values between $-1$ and $1$. The results are in Table 4.
Figure 3. Two parameter cost functionals for parameter sets (a) \((C, u_c)\), (b) \((C, \sigma_0)\) and (c) \((u_c, \sigma_0)\).

Table 1. Example 1: initial guesses used for identification experiments.

<table>
<thead>
<tr>
<th>initial guess</th>
<th>parameter values</th>
<th>initial guess</th>
<th>parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((0.7u_c, 0.7\sigma_0, 0.7C))</td>
<td>E</td>
<td>((1.3u_c, 0.7\sigma_0, 0.7C))</td>
</tr>
<tr>
<td>B</td>
<td>((0.7u_c, 0.7\sigma_0, 1.3C))</td>
<td>F</td>
<td>((1.3u_c, 0.7\sigma_0, 1.3C))</td>
</tr>
<tr>
<td>C</td>
<td>((0.7u_c, 1.3\sigma_0, 0.7C))</td>
<td>G</td>
<td>((1.3u_c, 1.3\sigma_0, 0.7C))</td>
</tr>
<tr>
<td>D</td>
<td>((0.7u_c, 1.3\sigma_0, 1.3C))</td>
<td>H</td>
<td>((1.3u_c, 1.3\sigma_0, 1.3C))</td>
</tr>
</tbody>
</table>

Table 2. Example 1: Results for the basic scheme (5) and the mollified scheme (7) with \(\eta = 3, 5\) and 8.

<table>
<thead>
<tr>
<th>IG</th>
<th>(j)</th>
<th>(p_j)</th>
<th>(E_j)</th>
<th>(e_o)</th>
<th>CPU [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic scheme (5)</td>
<td>A</td>
<td>2</td>
<td>(0.0697, 1.1219, 21.4706)</td>
<td>290</td>
<td>0.0651</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>4</td>
<td>(0.0696, 1.1111, 21.4700)</td>
<td>517</td>
<td>0.0741</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>5</td>
<td>(0.0697, 1.1324, 21.4700)</td>
<td>616</td>
<td>0.0564</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>3</td>
<td>(0.0696, 1.1252, 21.4699)</td>
<td>669</td>
<td>0.0623</td>
</tr>
</tbody>
</table>
### Table 3.

Example 2: Results for the basic scheme (5) and the mollified scheme (7) for different values of $\Delta x$ and $\eta$.

<table>
<thead>
<tr>
<th>$\Delta x/L$</th>
<th>Scheme</th>
<th># restarts</th>
<th>$E_j$</th>
<th>$e_{so} \pm \sigma$</th>
<th>CPU[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/128</td>
<td>(5)</td>
<td>387</td>
<td>53693</td>
<td>0.1400 ± 0.0329</td>
<td>49.17</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 3$</td>
<td>340</td>
<td>49274</td>
<td>0.1295 ± 0.0338</td>
<td>53.92</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 5$</td>
<td>333</td>
<td>50659</td>
<td>0.0909 ± 0.0315</td>
<td>50.91</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 8$</td>
<td>276</td>
<td>41263</td>
<td>0.0534 ± 3.72e-05</td>
<td>39.36</td>
</tr>
<tr>
<td>1/256</td>
<td>(5)</td>
<td>388</td>
<td>48842</td>
<td>0.0696 ± 0.0096</td>
<td>136.03</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 3$</td>
<td>398</td>
<td>53513</td>
<td>0.0739 ± 0.0081</td>
<td>160.74</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 5$</td>
<td>355</td>
<td>48280</td>
<td>0.0587 ± 0.0174</td>
<td>125.46</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 8$</td>
<td>326</td>
<td>48392</td>
<td>0.0431 ± 0.0190</td>
<td>114.35</td>
</tr>
<tr>
<td>1/512</td>
<td>(5)</td>
<td>334</td>
<td>38879</td>
<td>0.0312 ± 0.0038</td>
<td>440.91</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 3$</td>
<td>352</td>
<td>40416</td>
<td>0.0332 ± 0.0049</td>
<td>439.55</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 5$</td>
<td>346</td>
<td>41128</td>
<td>0.0278 ± 0.0057</td>
<td>352.47</td>
</tr>
<tr>
<td></td>
<td>(7), $\eta = 8$</td>
<td>374</td>
<td>47250</td>
<td>0.0195 ± 0.0072</td>
<td>335.24</td>
</tr>
</tbody>
</table>
Table 4. Example 3, $\Delta x = L/256$: results for the basic scheme (5) and the mollified scheme (7) for different values of $\varepsilon$ and $\eta$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Scheme</th>
<th># restarts</th>
<th>$E_I$</th>
<th>$\varepsilon_\infty \pm \sigma$</th>
<th>CPU[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>(5)</td>
<td>385</td>
<td>50310</td>
<td>0.0661 $\pm$ 0.0264</td>
<td>140.63</td>
</tr>
<tr>
<td>(7), $\eta = 3$</td>
<td>367</td>
<td>50489</td>
<td>0.0674 $\pm$ 0.0238</td>
<td>152.60</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 5$</td>
<td>368</td>
<td>52214</td>
<td>0.0533 $\pm$ 0.0256</td>
<td>135.85</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 8$</td>
<td>331</td>
<td>47153</td>
<td>0.0401 $\pm$ 0.0260</td>
<td>110.56</td>
<td></td>
</tr>
<tr>
<td>0.03</td>
<td>(5)</td>
<td>376</td>
<td>49025</td>
<td>0.0801 $\pm$ 0.0554</td>
<td>137.65</td>
</tr>
<tr>
<td>(7), $\eta = 3$</td>
<td>364</td>
<td>48447</td>
<td>0.0807 $\pm$ 0.0533</td>
<td>146.72</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 5$</td>
<td>374</td>
<td>50622</td>
<td>0.0760 $\pm$ 0.0528</td>
<td>132.10</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 8$</td>
<td>353</td>
<td>49136</td>
<td>0.0710 $\pm$ 0.0524</td>
<td>115.31</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>(5)</td>
<td>378</td>
<td>49716</td>
<td>0.1163 $\pm$ 0.0913</td>
<td>140.31</td>
</tr>
<tr>
<td>(7), $\eta = 3$</td>
<td>376</td>
<td>51246</td>
<td>0.1149 $\pm$ 0.0897</td>
<td>156.50</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 5$</td>
<td>359</td>
<td>50344</td>
<td>0.1129 $\pm$ 0.0902</td>
<td>131.10</td>
<td></td>
</tr>
<tr>
<td>(7), $\eta = 8$</td>
<td>332</td>
<td>49139</td>
<td>0.1084 $\pm$ 0.0915</td>
<td>115.54</td>
<td></td>
</tr>
</tbody>
</table>

3.3. Conclusions

According to Table 2, most of the identifications are successful and $\eta = 8$ seems to be the best choice. For the initial guess $A$ the method for $\eta = 5$ does not converge, but it does converge when started with initial guesses close to $A$. The results in Table 3, corresponding to Example 2, illustrate how by improving the spatial resolution (i.e., reducing $\Delta x$) the quality of the identification is increased.

Table 4 indicates that the level of noise influences the quality of the recovery but stability is never lost.

Summarizing, this parameter identification procedure yields good results for both the basic scheme and its mollified versions but the mollified approach returned advantages not only in CPU time (in s), but also in the error level, the sensitivity to the initial guess and the effect of noise in the data. This well-posed behavior was already suggested by the convex-shape of the cost functional (Figure 3.)

ACKNOWLEDGEMENTS

CDA and CEM acknowledge support by COLCIENCIAS project 1118-48925120 and by Universidad Nacional de Colombia, Vicerrectoría de Investigaciones, Projects 20201007076, 201010011107 and 20101009545. RB acknowledges support by Conicyt Anillo project ACT 1118 (ANANUM), Fondecyt project 1090456, BASAL project, CMM at Universidad de Chile, Centro de Investigación en Ingeniería Matemática (CIMAT) at Universidad de Concepción and Red Doctoral REDOC.CTA, project UCO 1202 at U. de Concepción.

REFERENCES


