A Variational Characterization of the Fucik Spectrum and Applications

Una caracterización variacional del espectro de Fucik y aplicaciones

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Dedicated to Professor Alan C. Lazer, our inspiring teacher.

Abstract. We characterize the Fucik spectrum (see [9]) of a class selfadjoint operators. Our characterization relies on Lyapunov-Schmidt reduction arguments. We use this characterization to establish the existence of solutions for a semilinear wave equation. This work has been motivated by the authors’ results in [4] where one dimensional second order ordinary differential equations are studied.

Key words and phrases. Fucik spectrum, Saddle point principle, Asymptotic behavior.

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Resumen. Se caracteriza el espectro de Fucik (véase [9]) de una clase de operadores autoadjuntos. Basamos esta caracterización en el método de reducción de Lyapunov-Schmidt. Usamos esta caracterización para demostrar la existencia de soluciones a una ecuación de onda semilineal. Este trabajo ha sido motivado por los resultados de los autores en [4] donde se estudian ecuaciones diferenciales ordinarias de segundo orden.

Palabras y frases clave. Espectro de Fucik, principio de puntos de silla, comportamiento asintótico.

1. Introduction

Let $\Omega$ be a measurable subset in $\mathbb{R}^n$ and $L$ a selfadjoint operator with discrete spectrum acting on $L^2(\Omega)$, the space of square integrable functions in $\Omega$. Examples of such operators are the Laplacian ($\Delta$) subject to Dirichlet or
Neumann boundary conditions in smooth bounded regions, and the wave operator \( (\square \equiv \partial_t^2 - \partial_{xx}) \) acting on 2\(\pi\)-periodic functions in the variable \(t\) that also satisfy the Dirichlet boundary condition \(u(0, t) = u(\pi, t) = 0\) (see [2]).

The Fucik spectrum of \(L, F\), is the set of pairs \((a, b)\) \(\in \mathbb{R}^2\) for which the equation
\[
Lu = au_+ - bu_- \quad \text{in} \quad \Omega
\]
has a non-zero solution, where \(u_+(x) = \max\{u(x), 0\}\), and \(u_-(x) = \max\{-u(x), 0\}\). This concept was introduced by S. Fucik in [9] in the context of differential equations.

**Remark 1.** If \(u \neq 0\) satisfies (1) then \(v = -u\) satisfies \(Lv = bv_+ - au_-.\) That is, \(F\) is symmetric with respect to the main diagonal in \(\mathbb{R}^2\). Since \(-L\) also has discrete spectrum, without loss of generality, we restrict our analysis to the case \(b > a\). Also by adding to \(L\) an adequate multiple of the identity one may assume \(b > a > 0\).

In order to establish our main result (Theorem 2 below) we recall the following global reduction principle (see [3]).

**Theorem 1.** Let \(H\) be a separable real Hilbert space. Let \(X, Y\) be closed subspaces such that \(H = X \oplus Y\), and \(J : H \to \mathbb{R}\) a functional of class \(C^1\). If there exists \(m > 0\) such that
\[
\langle \nabla J(x_1 + y) - \nabla J(x_2 + y), x_1 - x_2 \rangle \leq -m\|x_1 - x_2\|^2
\]
for all \(x_1, x_2 \in X, y \in Y\), then there exists a continuous function \(r : Y \to X\) such that

- \(J(y + r(y)) = \max\{J(y + x) \mid x \in X\}\).
- \(\tilde{J} : Y \to \mathbb{R}\) defined by \(\tilde{J}(y) = J(y + r(y))\) is of class \(C^1\).
- \(x + y\) is a critical point of \(J\) if and only if \(x = r(y)\) and \(y\) is critical point of \(\tilde{J}\).

We let \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\) and \(0 \geq \lambda_0 > \lambda_{-1} > \cdots > \lambda_{-n} > \cdots\) denote the eigenvalues of \(L\), and we assume that they do not have accumulation points in \(\mathbb{R}\). That is, if the set \(\{\lambda_i \mid i = 1, \ldots\}\) has infinitely many elements then \(\lim_{i \to \infty} \lambda_i = +\infty\). Similarly, if the set \(\{\lambda_{-i} \mid i = 1, \ldots\}\) has infinitely many elements then \(\lim_{i \to \infty} \lambda_{-i} = -\infty\).

Let \(\{\varphi_{j,k} \mid k = 1, 2, \ldots\}\) denote an orthonormal set of functions that span the set of eigenvectors corresponding to the eigenvalue \(\lambda_j\). We will denote by \(N(j)\) the multiplicity of the eigenvalue \(\lambda_j\), which need not be finite. We assume the set \(\{\varphi_{j,k} \mid j = 0, \pm 1, \ldots; k = 1, \ldots, N(j)\}\) to be complete in \(L^2(\Omega)\). Let \(H\) denote the subspace of \(L^2(\Omega)\) of elements of the form
such that
\[ \sum_{j=-\infty, k=1}^{\infty, N(j)} |\lambda_j| (a_{j,k})^2 < \infty. \]

It is easily seen that \( H \) is a Hilbert space under the inner product
\[ \left\langle \sum_{j=-\infty, k=1}^{\infty, N(j)} a_{j,k} \phi_{j,k}, \sum_{j=-\infty, k=1}^{\infty, N(j)} b_{j,k} \phi_{j,k} \right\rangle_1 = \sum_{j=-\infty, k=1}^{\infty, N(j)} (1 + |\lambda_j|) a_{j,k} b_{j,k}. \]

We denote by \( \| \cdot \|_1 \) the norm defined by the inner product \( \langle \cdot, \cdot \rangle_1 \).

We let \( g_{a,b} \equiv g : \mathbb{R} \to \mathbb{R} \) be given by
\[ g(t) = at \quad \text{for} \quad t \geq 0 \quad \text{and} \quad g(t) = bt \quad \text{for} \quad t \leq 0. \]

For \( u \) as in (3) and \( v = \sum_{j=-\infty, k=1}^{\infty, N(j)} b_{j,k} \phi_{j,k} \) we define
\[ B(u, v) = \sum_{j=-\infty, k=1}^{\infty, N(j)} \lambda_j a_{j,k} b_{j,k}. \]

With \( u \) as in (3), let \( J : H \to \mathbb{R} \) be defined by
\[ J_{a,b}(u) \equiv J(u) = (1/2) \left( B(u, u) - \int_{\Omega} u(x) g(u(x)) \, dx \right). \]

Note that if \( L(u) \in L^2(\Omega) \), i.e. if \( \sum_{j=-\infty, k=1}^{\infty, N(j)} |\lambda_j|^2 |(a_{j,k})|^2 < \infty \), then
\[ B(u, v) = \langle L(u), v \rangle_0, \]
where \( \langle \cdot, \cdot \rangle_0 \) denotes the usual inner product in \( L^2(\Omega) \). Standard calculations prove that, for \( u \) as in (3) and \( v = \sum_{j=-\infty, k=1}^{\infty, N(j)} b_{j,k} \phi_{j,k} \),
\[ \langle \nabla J(u), v \rangle_1 = \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} = \sum_{j=-\infty, k=1}^{\infty, N(j)} \lambda_j a_{j,k} b_{j,k} - \int_{\Omega} g(u(x)) v(x) \, dx \]
\[ = B(u, v) - \int_{\Omega} g(u(x)) v(x) \, dx. \]
Let Lemma 1.

In the next two lemmas we prove that the functions $H$ depend continuously on $(a, b)$. For each pair $(a, b)$ of positive numbers that converges to +1. Therefore (2) is satisfied and, hence,

$$\{\parallel \lambda x - y \parallel \leq a \parallel x_1 - x_2 \parallel_0 \leq -m \parallel x_1 - x_2 \parallel^2_1, (11)$$

where $m \equiv m(a) = \inf \{ (a - \lambda_i)/(1 + |\lambda_i|) \mid i \leq j \} > 0$. Note that $m > 0$ since $(a - \lambda_i)/(1 + |\lambda_i|)$ is either finite set of positive numbers or a sequence of positive numbers that converges to +1. Therefore (2) is satisfied and, hence, for each pair $(a, b)$ there exists a continuous function $r_{a,b} \equiv \lambda$ satisfying the properties in Theorem 1. For future reference, and using that $g$ is homogeneous of degree one, we note that for any $x \in X$ and $\lambda > 0$ we have

$$0 = \lambda \left( B(r(y), x) - \int_{\Omega} xg(y + r(y)) \, d\xi \right)$$

$$= B(\lambda r(y), x) - \int_{\Omega} xg(\lambda y + \lambda r(y)) \, d\xi. (12)$$

Hence

$$r(\lambda y) = \lambda r(y) \quad \text{for any} \quad \lambda > 0. (13)$$

In the next two lemmas we prove that the functions $r_{a,b}$ are compact and depend continuously on $(a, b)$.

**Lemma 1.** Let $N(l) < \infty$ for all $l > j$. If $\{y_n\}^\infty_n$ converges weakly to $\overline{y}$ then $\{r_{a,b}(y_n)\}^\infty_n$ contains a subsequence that converges to $r_{a,b}(\overline{y})$.

**Proof.** For the sake of simplicity in the notation, throughout this proof we write $r$ for $r_{a,b}$, and $g$ for $g_{a,b}$. Let $\{y_n\}^\infty_n$ converge weakly to $\overline{y}$. Since

$$m \parallel r(y_n) \parallel^2 \leq -\langle \nabla J_{a,b}(y_n + r(y_n)) - \nabla J_{a,b}(y_n), r(y_n) \rangle_1$$

$$= \langle \nabla J_{a,b}(y_n), r(y_n) \rangle_1$$

$$= -\int_{\Omega} g(y_n) r(y_n) \, d\xi$$

$$\leq b \parallel y_n \parallel_0 \parallel r(y_n) \parallel_0, (14)$$

the sequence $\{r(y_n)\}$ is bounded. Since $N(l) < \infty$ for all $l > j$, the imbedding of $Y$ into $L^2(\Omega)$ is compact. Thus, without loss of generality, we may assume

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that \( \{y_n\} \) converges in \( L^2(\Omega) \) to \( \bar{y} \). From the definition of \( r \) we have

\[
(a - \lambda_j)\|r(y_n) - r(y_m)\|_0^2 + a\|y_n - y_m\|_0^2 \\
\leq -B(r(y_n) - r(y_m), r(y_n) - r(y_m)) \\
+ \int_\Omega (g(y_n + r(y_n)) - g(y_m + r(y_m))) (y_n + r(y_n) - y_m - y_m) \, d\zeta \\
= \int_\Omega (g(y_n + r(y_n)) - g(y_m + r(y_m))) (y_n - y_m) \, d\zeta. \tag{15}
\]

Since \( \{y_n\} \) is a Cauchy sequence in \( L^2(\Omega) \) and \( \{g(y_n + r(y_n))\} \) is bounded in \( L^2(\Omega) \), the last term in (15) tends to zero, which proves that \( \{r(y_n)\} \) is a Cauchy sequence in \( L^2(\Omega) \). Let \( z \) be the limit of \( \{r(y_n)\} \) in \( L^2(\Omega) \). Hence \( g(y_n + r(y_n)) \) converges to \( g(\bar{y} + z) \), and

\[
0 = B(z, x) - \int_\Omega g(\bar{y} + z) x \, d\xi \tag{16}
\]

for any \( x \in X \). By the uniqueness of \( r(\bar{y}) \) we conclude that \( z = r(\bar{y}) \), which proves the lemma.

**Lemma 2.** If \( \{(a_n, b_n)\}_n \) converges to \((a, b)\), \( b > a, b_n > a_n \) and \( a, a_n \in (\lambda_j, \lambda_{j+1}) \), then \( \{r_{a_n, b_n}(y)\}_n \) converges to \( r_{a, b}(y) \) for each \( y \in Y \), i.e., \( r \) depends continuously on \((a, b)\).

**Proof.** Letting \( z = r_{a_n, b_n}(y) - r_{a, b}(y) \), from the definition of \( r \) we have

\[
0 = B(z, z) - \int_\Omega (g_{a_n, b_n}(y + r_{a_n, b_n}(y)) - g_{a, b}(y + r_{a, b}(y))) z \, d\xi \\
= B(z, z) - \int_\Omega (g_{a_n, b_n}(y + r_{a_n, b_n}(y)) - g_{a_n, b_n}(y + r_{a, b}(y))) z \, d\xi \\
- \int_\Omega (g_{a_n, b_n}(y + r_{a, b}(y)) - g_{a, b}(y + r_{a, b}(y))) z \, d\xi. \tag{17}
\]

From (11), (17), and the fact that \( (g_{a_n, b_n}(t) - g_{a, b}(t))/t \) converges to 0 uniformly for \( t \in \mathbb{R} \) as \( n \to \infty \), we have

\[
m\|z\|_1^2 \leq \|g_{a_n, b_n}(y + r_{a, b}(y)) - g_{a, b}(y + r_{a, b}(y))\|_0 \|z\|_0. \tag{18}
\]

Hence, given \( \epsilon > 0 \) there exists \( N \) such that if \( n \geq N \) then

\[
m\|z\|_1 \leq \|g_{a_n, b_n}(y + r_{a, b}(y)) - g_{a, b}(y + r_{a, b}(y))\|_0 \leq \epsilon, \tag{19}
\]

which proves the lemma.

Our main result is the following.
Theorem 2. If $a \in (\lambda_j, \lambda_{j+1})$, $N(l) < \infty$ for $l \geq j + 1$, and $b_1(a) \equiv b_1 = \sup \{b \geq a \mid \tilde{J}_{\alpha,\beta}(y) = J_{\alpha,\beta}(y + r_{\alpha,\beta}(y)) > 0 \text{ for all } \beta \in (a,b), y \in Y - \{0\}\}$, then

a) $(a, b_1)$ is in the Fucik spectrum when $b_1 < +\infty$.

b) If $b \in [a, b_1)$ then $(a, b)$ is not in the Fucik spectrum.

c) For $b > a$, $(a,b)$ is in the Fucik spectrum if and only if the restriction of $\tilde{J}_{\alpha,b}$ to $\{y \in Y \mid \|y\|_1 = 1\}$ has a critical point on $\{y \in Y \mid \|y\|_1 = 1, \tilde{J}_{\alpha,b} = 0\}$.

d) The function $b_1 : (\lambda_j, \lambda_{j+1}) \to [0, +\infty]$, $a \mapsto b_1(a)$ is non-increasing and continuous.

Remark 2. In general, even when $X$ is finite dimensional, $b_1(a)$ need not be finite for all $a \in (\lambda_j, \lambda_{j+1})$. For example, it is easily seen that for $a \in (0, 0.25]$ the equation

$$-u'' = au_+ - bu_- \quad \text{in} \quad (0, \pi), \quad u'(0) = u'\pi) = 0 \quad (20)$$

has no non-trivial solution. That is, $b_1(a) = +\infty$ for all $a \in (0, 0.25]$. In this case $\lambda_0 = 0$ and $\lambda_1 = 1$.

In Lemma 7 we present a sufficient condition for $b_1(a)$ to be finite for all $a \in (\lambda_j, \lambda_{j+1})$. See Remark 3 for an application of Lemma 7.

For recent results on variational characterizations of the Fucik spectrum the reader is referred to [10] and [11] where a different variational characterization of the Fucik spectrum is provided. Unlike the results of [10] and [11], Theorem 2 includes operators $L$ with infinitely many positive and infinitely many negative eigenvalues which may have infinite multiplicity. This allows for applications to non-elliptic problems such as the wave equation (21) below. Theorem 2 was motivated by the authors’ work in [4] where the existence of periodic solutions for a semilinear ordinary differential equation is established using that the corresponding potential is asymptotically equal to $ug_{a,b}(u)/2$ with $(a,b)$ not in the Fucik spectrum. For other results on the Fucik spectrum the reader is referred to [1, 6, 5, 8, 7, 12]; none of which study (1) in the generality presented here.

As an application of Theorem 2 we establish the existence of weak solutions for the semilinear wave equation

$$u_{tt}(x, t) - u_{xx}(x, t) = h\big(u(x, t)\big) + p(x, t), \quad \text{for } x \in (0, \pi), t \in \mathbb{R}$$

$$u(x, t) = u(x, t + 2\pi), \quad \text{for } x \in (0, \pi), t \in \mathbb{R}, \quad (21)$$

$$u(0, t) = u(\pi, t) = 0, \quad \text{for } t \in \mathbb{R}.$$
where \( h : \mathbb{R} \to \mathbb{R} \) is a continuous function, \( p \in L^2((0, \pi) \times (0, 2\pi)) \), and \( p \) is \( 2\pi \)-periodic in the variable \( t \). The spectrum of \( \Box = \partial_{tt} - \partial_{xx} \), D’Alembert’s operator is given by \( \{ k^2 - j^2 | k = 1, 2, \ldots, j = 0, 1, \ldots \} \). Thus \( \lambda_0 = 0 \), \( \lambda_1 = 1 \). We assume that \( h'(t) \geq \epsilon > 0 \) for all \( t \in \mathbb{R} \). We let \( H(s) = \int_0^s h(t) \, dt \), and assume that that there exists positive real numbers \( a, b \) such that

\[
\limsup_{s \to +\infty} \frac{2H(s)}{s^2} = a, \quad \limsup_{s \to -\infty} \frac{2H(s)}{s^2} = b,
\]

\[a \in (0, 1) \quad \text{and} \quad b \in (a, b_1(a)), \tag{23}\]

where \( b_1 \equiv b_1(a) \) is as in Theorem 2.

Using Theorem 2 we prove the following result.

**Theorem 3.** *If \((22)\) and \((23)\) hold, then the equation \((21)\) has a weak solution.*

For the version of Theorem 3 to ordinary differential equations see [4]. The reader is invited to compare this result with Theorem 1 of [2] where an existence result for \((21)\) is established when \((a, b)\) is restricted to the rectangle \((0, 1) \times (0, 1)\).

### 2. Proof of Theorem 2

Without loss of generality we may assume that \( a > 0 \).

First we note that \( b_1 \geq \lambda_{j+1} \). In fact, if \( b \in [a, \lambda_{j+1}] \) then, for \( y \neq 0 \),

\[
\tilde{J}_{a,b}(y) = J_{a,b}(y + r(y)) \\
\geq J_{a,b}(y) \\
= B(y, y) - \int_{\Omega} y(\xi)g_{a,b}(y(\xi)) \, d\xi \\
\geq B(y, y) - b \int_{\Omega} y^2(\xi) \, d\xi \\
\geq \frac{\lambda_{j+1} - b}{\lambda_{j+1}} B(y, y) \\
> 0.
\]

Next we relate the Fucik spectrum of \( L \) with the critical points of \( J_{a,b} \).

**Lemma 3.** *The pair \((a, b)\) \(\in\) \( F \) if and only if \( J_{a,b} \) has a nonzero critical point.*

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Proof. If \( u \neq 0 \) is a solution to (1) then multiplying (1) by \( v \) and using (9) we have

\[
0 = \langle L(u), v \rangle_0 - \int_\Omega g_{a,b}(u)v \, d\zeta \\
= B(u, v) - \int_\Omega g_{a,b}(u)v \, d\zeta \\
= \langle \nabla J_{a,b}(u), v \rangle_1.
\]

(25)

Thus \( u \) is a critical point of \( J_{a,b} \).

On the other hand, if \( u = \sum_{j=-\infty}^{\infty, N(j)} \sum_{k=1}^{a_j, k} a_{j,k} \phi_{j,k} \neq 0 \) is a critical point of \( J_{a,b} \) letting

\[
\begin{align*}
  u_- &= \sum_{j=-l,k=1}^{0, \min\{N(j), l\}} a_{j,k} \phi_{j,k} \quad \text{and} \\
  u_+ &= \sum_{j=1,k=1}^{l, \min\{N(j), l\}} a_{j,k} \phi_{j,k},
\end{align*}
\]

(26)

we see that \( L(u_-), L(u_+) \in H \) and \( \{u_- + u_+\} \) converges to \( u \) in \( H \), hence in \( L^2(\Omega) \). Thus \( 0 = \langle \nabla J_{a,b}(u), L(u_+) - L(u_-) \rangle_1 \). This and the fact that \( L(u_+) \) and \( L(u_-) \) are in orthogonal subspaces give

\[
\begin{align*}
  &\|L(u_+) + L(u_-)\|_0^2 = \|L(u_+) - L(u_-)\|_0^2 \\
  &= \sum_{j=-l,k=1}^{0, \min\{N(j), l\}} \lambda_{j,k}^2 a_{j,k}^2 + \sum_{j=1,k=1}^{l, \min\{N(j), l\}} \lambda_{j,k}^2 a_{j,k}^2 \\
  &= B(u, L(u_+) - L(u_-)) \\
  &= \int_\Omega (L(u_+) - L(u_-)) g_{a,b}(u) \\
  &\leq \|L(u_+) - L(u_-)\|_0 g_{a,b}(u) \|_0.
\end{align*}
\]

(27)

Thus \( \{\|L(u_+) + L(u_-)\|_0^2\}_l \) is bounded, which implies that \( \{L(u_- + u_+)\}_l \) defines a Cauchy sequence in \( L^2(\Omega) \). Since \( L \) si assumed to be selfadjoint, hence closed, \( u \) is in the domain of \( L \). That is \( L(u) \in L^2(\Omega) \). Hence for all \( v \in L^2(\Omega) \)

\[
\int_\Omega v g_{a,b}(u) = B(u, v) = \langle L(u), v \rangle_0.
\]

(28)

Thus \( L(u) = g_{a,b}(u) = au_+ - bu_- \), which proves the lemma. \( \square \)

Lemma 4. If \( b \in [a, b_1) \) then \( (a, b) \notin \mathcal{F} \).
Proof. By the definition of \( b_1 \), if \( b \in [a,b_1) \) then \( \tilde{J}_{a,b}(y) > 0 \) for any \( y \in Y \) with \( \|y\| = 1 \). Hence
\[
\langle \nabla J_{a,b}(y + r(y)), y + r(y) \rangle_1 \\
= B(y + r(y), y + r(y)) - \int_{\Omega} (y + r(y))g_{a,b}(y + r(y)) \, d\zeta \\
= 2J_{a,b}(y + r(y)) \\
= 2\tilde{J}_{a,b}(y) \\
> 0.
\]

Thus, by Theorem 1, \( \nabla J(y + x) \neq 0 \) for \( y + x \neq 0 \), which proves the lemma. \( \Box \)

Lemma 5. If \( b_1(a) < \infty \) and \( N(l) < \infty \) for all \( l \geq j + 1 \), then there exists \( y_0 \in Y \) with \( \|y_0\|_1 = 1 \) and such that
\[
\tilde{J}_{a,b_1}(y_0) = 0 = \min \{ \tilde{J}_{a,b_1}(y) \mid \|y\|_1 = 1 \}.
\]

Proof. By the definition of \( b_1 \) there exists a sequence \( \{\beta_i\}_i \) converging to \( b_1 \) and a sequence \( \{y_i\}_i \) with \( \|y_i\|_1 = 1 \) such that \( \tilde{J}_{a,\beta_i}(y_i) \leq 0 \). Using again that \( \lambda_j \to +\infty \) as \( j \to \infty \), one sees that \( \{y_i\} \) has a subsequence that converges strongly in \( L^2(\Omega) \). For the sake of simplicity in the notations we denote by \( \{y_i\} \) such a subsequence and denote by \( \tilde{y} \) its weak limit in \( H \) which is its strong limit in \( L^2(\Omega) \). Since, by the definition of \( X,Y \), the functional \( J_{a,\beta} \), satisfies (2) we have
\[
m\|r_{a,\beta_i}(y_i)\|^2 \leq -\langle \nabla J_{a,\beta_i}(y_i + r_{a,\beta_i}(y_i)) - \nabla J_{a,\beta_i}(y_i), r_{a,\beta_i}(y_i) \rangle_1 \\
= \langle \nabla J_{a,\beta_i}(y_i), r_{a,\beta_i}(y_i) \rangle_1 \\
= -\int_{\Omega} r_{a,\beta_i}(y_i)g_{a,\beta_i}(y_i) \, d\zeta.
\]

Since \( |g_{a,\beta_i}(t)| \leq c|t| \) for some constant \( c \) independent of \( i \) and \( t \), we see that \( \{r_{a,\beta_i}(y_i)\} \) is bounded in \( H \). Let us also see that \( \{r_{a,\beta_i}(y_i)\}_i \) is also a Cauchy sequence in \( H \). In fact, letting \( z_k = r_{a,b_k}(y_k) \) we have
\[
m\|z_i - z_j\|^2 \leq -\langle \nabla J_{a,\beta_i}(y_i + z_i) - \nabla J_{a,\beta_i}(y_i + z_j), z_i - z_j \rangle_1 \\
= B(z_j, z_i - z_j) - \int_{\Omega} (z_i - z_j)(g_{a,\beta_i}(y_i + z_j)) \, d\zeta \\
= \int_{\Omega} (z_i - z_j)(g_{a,\beta_i}(y_j + z_j) - g_{a,\beta_i}(y_i + z_j)) \, d\zeta \\
= \int_{\Omega} (z_i - z_j)(g_{a,\beta_i}(y_j + z_j) - g_{a,\beta_i}(y_i + z_j)) \, d\zeta \\
\quad \quad \quad \quad + \int_{\Omega} (z_i - z_j)(g_{a,\beta_i}(y_i + z_j) - g_{a,\beta_i}(y_i + z_j)) \, d\zeta \\
\equiv I_1 + I_2.
\]

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An elementary calculation shows that $|g_{a,\beta_i}(s) - g_{a,\beta_j}(t)| \leq \beta_j |s - t|$ for any $s, t \in \mathbb{R}$. Hence $\| (g_{a,\beta_i}(y_i + z_i) - g_{a,\beta_j}(y_j + z_j)) \|_0$ converges to 0 as $i, j$ tend to infinity. This and the fact that $\{z_i\}_i$ is bounded in $L^2(\Omega)$ (see (30)) prove that the integral $I_1$ in (31) converges to zero as $i, j \to +\infty$. The term $I_2$ converges to zero as $i, j \to +\infty$ because $\{z_i\}_i$ is bounded in $L^2(\Omega)$ and $\{\beta_i\}_i$ converges. Let $\lim z_i = z \in X$. Therefore, for any $x \in X$, we have

$$0 = \lim_{i \to \infty} \left( B(z_i, x) - \int_\Omega xg_{a,\beta_i}(y_i + z_i) \, d\zeta \right) = B(z, x) - \int_\Omega xg_{a,b_1}(\tilde{y} + z) \, d\zeta,$$

which implies that $z = r_{a,b_1}(\tilde{y})$.

From (30) we see that if $\tilde{y} = 0$, $\lim_{i \to \infty} ||z_i|| = 0$. On the other hand, since $\tilde{J}_{a,\beta_i}(y_i) \leq 0$ we have

$$0 \geq \limsup_{i \to \infty} 2\tilde{J}_{a,\beta_i}(y_i) = \lim_{i \to \infty} \left( B(y_i, y_i) + B(z_i, z_i) - \int_\Omega (y_i + z_i)g_{a,\beta_i}(y_i + z_i) \, d\zeta \right),$$

which contradicts that $B(y_i, y_i) \geq (\lambda_{j+1}/(\lambda_{j+1} + 1))||y_i||^2 = \lambda_{j+1}/(\lambda_{j+1} + 1) > 0$ and $\lim_{i \to \infty} (B(z_i, z_i) - \int_\Omega (y_i + z_i)g_{a,\beta_i}(y_i + z_i) \, d\zeta) = 0$. Thus $\tilde{y} \neq 0$.

From the definition of $r$ we have $0 = B(z_i, z_i) - \int_\Omega z_ig_{a,\beta_i}(y_i + z_i) \, d\zeta$. Thus

$$2\tilde{J}_{a,b_1}(\tilde{y}) = B(\tilde{y}, \tilde{y}) + B(r(\tilde{y}), r(\tilde{y})) - \int_\Omega (\tilde{y} + r(\tilde{y}))g_{a,b_1}(\tilde{y} + r(\tilde{y})) \, d\zeta$$

$$\leq \liminf_{i \to \infty} B(y_i, y_i) - \int_\Omega \tilde{y}g_{a,b_1}(\tilde{y}) \, d\zeta$$

$$= \liminf_{i \to \infty} \left( B(y_i, y_i) - \int_\Omega y_ig_{a,\beta_i}(y_i + z_i) \, d\zeta \right)$$

$$\leq 0.$$

Since $\tilde{J}(\lambda y) = J(\lambda y + r(\lambda y)) = \lambda^2 J(y + r(y))$ we have $\tilde{J}_{a,b_1}((1/||\tilde{y}||)\tilde{y}) \leq 0$, which proves that

$$\inf \{ \tilde{J}_{a,b_1}(y) \mid ||y||_1 = 1 \} \leq 0.$$  

(35)

Assuming that $\tilde{J}_{a,b_1}(y) < 0$ for some $y$ with $||y||_1 = 1$, by the continuity of $r$ for $\epsilon > 0$ close to zero we have $\tilde{J}_{a,b_1}(y) < 0$. Since this contradicts the definition of $b_1$ we have $\inf \{ \tilde{J}_{a,b_1}(y) \mid ||y||_1 = 1 \} = 0$. Taking $y_0 = (1/||\tilde{y}||_1)\tilde{y}$ the lemma is proven. 

Lemma 6. For $y_0$ as in Lemma 5 we have $\nabla \tilde{J}(y_0) = 0$. ☐

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Proof. Since $y_0$ is a critical point of $\tilde{J}_{a,b_1}$ restricted to the unit sphere in $H$, by the Lagrange multipliers rule there exists $\lambda \in \mathbb{R}$ such that $\nabla \tilde{J}_{a,b_1}(y_0) = \lambda y_0$. Thus

\begin{equation}
0 = 2\tilde{J}_{a,b_1}(y_0)
= B(y_0, y_0) + B(r(y_0), r(y_0)) - \int_{\Omega} (y_0 + r(y_0)) g_{a,b_1}(y_0 + r(y_0)) \, d\zeta
= \langle \nabla \tilde{J}_{a,b_1}(y_0), y_0 \rangle_1
= \lambda \langle y_0, y_0 \rangle_1,
\end{equation}

which implies that $\lambda = 0$ since $\|y_0\|_1 = 1$. Hence $y_0$ is a critical point of $\tilde{J}_{a,b_1}$ which proves the lemma.

Proof. (Theorem 2)

• Part a) of Theorem 2 follows from Lemmas 5-6.

• Part b) was proved in Lemma 4.

• Since also $\langle \nabla J_{a,b}(x + y), x + y \rangle = 2J(x + y) = \tilde{J}(y)$ we have that the critical points of $J$ are the critical points of $\tilde{J}$ restricted to the unit sphere with $\tilde{J}(y) = 0$, which proves part c).

• Now we prove part d). Let $\tilde{y}$ be such that

\begin{equation}
0 = \tilde{J}_{a,b_1}(\tilde{y}) = J_{a,b_1}(\tilde{y} + r_{a,b_1}(\tilde{y}))
= \min \left\{ J_{a,b_1}(y + r_{a,b_1}(y)) \mid y \in Y, \|y\|_1 = 1 \right\}.
\end{equation}

Since $L(\tilde{y} + r_{a,b_1}(\tilde{y})) = g_{a,b_1}(\tilde{y} + r_{a,b_1}(\tilde{y}))$ and $a$ is not an eigenvalue of $L$, $\tilde{y} + r_{a,b_1}(\tilde{y})$ is not a positive function. Hence, letting $G_{a,b}(u) = (1/2)u g_{a,b}(u)$, for any $\delta > 0$ we have

\begin{equation}
2\tilde{J}_{a,b_1}(\tilde{y}) + \delta \langle \tilde{y} \rangle
= \max_{x \in X} \left\{ B(x + \tilde{y}, x + \tilde{y}) - \int_{\Omega} G_{a,b_1}(x + \tilde{y}) \right\}
= \max_{x \in X} \left\{ B(x + \tilde{y}, x + \til{y}) - \int_{\Omega} G_{a,b_1}(x + \til{y}) - \int_{\Omega} G_{0,\delta}(x + \til{y}) \right\}
= B(r_{a,b_1}(\til{y}) + \til{y}, r_{a,b_1}(\til{y}) + \til{y})
- \int_{\Omega} G_{a,b_1}(r_{a,b_1}(\til{y}) + \til{y}) - \int_{\Omega} G_{0,\delta}(r_{a,b_1}(\til{y}) + \til{y})
< 0,
\end{equation}

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where we have used that if \( r_{a,b_1(a)+\delta} (y) \neq r_{a,b_1(a)} (\tilde{y}) \), then
\[
B(r_{a,b_1(a)+\delta} (y) + \tilde{y}, r_{a,b_1(a)+\delta} (\tilde{y}) + \tilde{y})
- \int_{\Omega} G_{a,b_1(a)} (r_{a,b_1(a)+\delta} (y) + \tilde{y}) \, d\zeta < 0, \tag{39}
\]
while if \( r_{a,b_1(a)+\delta} (y) = r_{a,b_1(a)} (\tilde{y}) \) then \( - \int_{\Omega} G_{a,b_1(a)} (r_{a,b_1(a)+\delta} (y) + \tilde{y}) \, d\zeta < 0 \) since \( r_{a,b_1(a)} (\tilde{y}) + \tilde{y} \) is not a positive function.

Arguing as in (38) we see that for any \( \delta \in (0, \lambda_j+1 - a) \),
\[
\tilde{J}_{a+\delta,b_1(a)} (\tilde{y}) \leq 0. \tag{40}
\]
Hence \( b_1(a+\delta) \leq b_1(a) \), which proves that \( b_1 \) is a non-increasing function.

Let \( \{a_n\}_{n} \) be a sequence in \( (\lambda_j, \lambda_{j+1}) \) converging to \( a \). Suppose that \( b_1(a_n) \leq b_1(a) - \delta \) for some \( \delta > 0 \). By the definition of \( b_1(a_n) \) there exists \( y_n \in Y \) with \( \| y_n \|_1 = 1 \) such that \( \tilde{J}_{a_n,b_1(a_n)} (y_n) = 0 \). Since \( Y \) is compactly imbedded in \( L^2(\Omega) \), we may assume without loss of generality that \( \{y_n\} \) converges weakly to \( \overline{y} \) in \( Y \) and that \( \{y_n\} \) converges strongly to \( \overline{y} \) in \( L^2(\Omega) \). Since
\[
B(y_n - y_m, y_n - y_m)
= \int_{\Omega} (y_n - y_m) (g_n (y_n + r_n (y_n)) - g_m (y_m + r_m (y_m))) \, d\zeta, \tag{41}
\]
where \( g_n = g_{a_n,b_1(a_n)} \), \( r_n = r_{a_n,b_1(a_n)} \), similarly \( g_m, r_m \). Hence \( \{y_n\}_{n} \) converges strongly to \( \overline{y} \) in \( H \). Let \( c \leq b_1(a) - \delta \) be a limit point of \( \{b_1(a_n)\}_{n} \).

Without loss of generality we may assume that \( \{b_1(a_n)\}_{n} \) converges to \( c \).
Thus
\[
\tilde{J}_{a,c} (\overline{y}) = J_{a,c} (\overline{y} + r_{a,c} (\overline{y}))
= \lim_{n \to \infty} J_{a_n,b_1(a_n)} (\overline{y} + r_{a_n,b_1(a_n)} (\overline{y}))
= \lim_{n \to \infty} J_{a_n,b_1(a_n)} (y_n + r_{a_n,b_1(a_n)} (y_n))
= 0,
\]
which contradicts the definition of \( b_2(a) \). Hence
\[
\liminf_{t \to a} b_1(t) \geq b_1(a). \tag{43}
\]

From (38) we have
\[
\limsup_{n \to \infty} \tilde{J}_{a_n,b_1(a)+\delta} (\overline{y}) = \limsup_{n \to \infty} J_{a_n,b_1(a)+\delta} (\overline{y} + r_{a_n,b_1(a)+\delta} (\overline{y}))
= J_{a,b_1(a)+\delta} (\overline{y} + r_{a,b_1(a)+\delta} (\overline{y}))
= \tilde{J}_{a,b_1(a)+\delta} (\overline{y}) < 0. \tag{44}
\]
Hence, for $n$ sufficiently large, $b_1(a_n) \leq b_1(a) + \delta$. Since $\delta > 0$ is arbitrary,

$$\limsup_{t \to a} b_1(t) \leq b_1(a).$$

(45)

From (43) and (45) we conclude that $b_1$ is continuous, which concludes the proof of Theorem 2.

3. A Sufficient Condition for $b_1(a) < \infty$

**Lemma 7.** If $Y \setminus \{0\}$ contains a non-negative function then $b_1(a) < +\infty$ for all $a \in (\lambda_k, \lambda_{k+1})$.

**Proof.** Let $y \in Y \setminus \{0\}$ be a non-negative function. Assuming that

$$\inf_{x \in X} \int_\Omega ((-y + x)_-)^2 = 0,$$

there exists a sequence $\{x_k\} \in X$ such that

$$0 = \inf_{x \in X} \int_\Omega ((-y + x)_-)^2 = \lim_{k \to \infty} \int_\Omega ((-y + x_k)_-)^2.$$

(46)

Writing $2x_k = (-y + x_k) + (x_k + y) = (-y + x_k)_+ - (-y + x_k)_- + (y + x_k)$, and using (46) we have

$$0 = 2 \int_\Omega x_k y = \lim_{k \to \infty} \int_\Omega ((-y + x_k)_+ + (y + x_k)y) d\zeta \geq \|y\|_0^2 > 0.$$

(47)

This contradiction proves that $c = \inf_{x \in X} \int_\Omega ((-y + x)_-)^2 > 0$. Now, for any $x \in X$,

$$2J(-y + x) = B(-y, -y) - a\|y\|_0^2 + B(x, x) - a\|x\|_0^2$$

$$- (b - a) \int_\Omega ((-y + x)_-)^2 d\xi$$

$$\leq B(y, y) - a\|y\|_0^2 - c(b - a)$$

$$< 0,$$

for $b > a + (B(y, y) - a\|y\|_0^2)/c$. Hence $\tilde{J}(-y) = \max\{J(-y + x) \mid x \in X\} < 0$ and $b_1(a) \leq a + (B(y, y) - a\|y\|_0^2)/c < +\infty$, which proves the lemma.

4. Proof of Theorem 3

Let $W = (0, \pi) \times (0, 2\pi)$ and $H$ be the vector space of elements $u \in L^2(W)$ with

$$u(x, t) = \sum_{k=1, j=0}^{\infty, \infty} a_{k,j} \sin(kx) \cos(jt) + b_{k,j} \sin(kx) \sin(jt)$$

(49)
and
\[
\sum_{k=1, j=0}^{\infty} (1 + |j^2 - k^2|)(a_{k,j}^2 + b_{k,j}^2) < \infty. 
\]  
(50)

This vector space is a Hilbert space under the inner product defined by
\[
\langle u, v \rangle_1 = \sum_{k=1, j=0}^{\infty} (1 + |j^2 - k^2|)(a_{k,j} \alpha_{k,j} + b_{k,j} \beta_{k,j}) \delta_{kj}, 
\]  
(51)

where \( \delta_{k0} = \pi^2 \), \( \delta_{kj} = \pi^2/2 \) for \( j > 0 \), \( u \) is as in (49), and \( v \) is given by
\[
v(x, t) = \sum_{k=1, j=0}^{\infty} \alpha_{k,j} \sin(kx) \cos(jt) + \beta_{k,j} \sin(kx) \sin(jt). 
\]  
(52)

For \( u, v \) as above, let
\[
B(u, v) = \sum_{k=1, j=0}^{\infty} \delta_{kj}(k^2 - j^2)(a_{k,j} \alpha_{k,j} + b_{k,j} \beta_{k,j}). 
\]  
(53)

Note that if \( u \) is a function of class \( C^2 \) and \( \Box u \in L^2(\Omega) \) then
\[
B(u, v) = \langle \Box u, v \rangle_0. 
\]  
(54)

where \( \Gamma(t) = \int_0^t h(s) \, ds \). We say that \( u \in H \) is a weak solution to (21) if \( u \) is a critical point of \( I \). Let \( X \) be the closure of the subspace of \( H \) generated by functions of the type \( \sin(kx) \cos(jt) \), \( \sin(kx) \sin(jt) \) such that \( k^2 - j^2 \leq 0 \), and \( Y \) the closure of the subspace of \( H \) generated by functions of the type \( \sin(kx) \cos(jt) \), \( \sin(kx) \sin(jt) \) such that \( k^2 - j^2 \geq 1 \). A straightforward calculation shows that
\[
\langle \nabla I(u), v \rangle = B(u, v) - \int_W (h(u) + p) v \, dx \, dt. 
\]  
(55)

Since \( B(z, z) \leq 0 \) for any \( z \in X \), for \( y \in Y, z_1, z_2 \in X \) we have
\[
\langle \nabla I(y + z_1) - \nabla I(y + z_2), z_1 - z_2 \rangle = \\
B(z_1 - z_2, z_1 - z_2) - \int_W (h(y + z_1) - h(y + z_2))(z_1 - z_2) \, dx \, dt \\
\leq -\epsilon \|z_1 - z_2\|_1^2, 
\]  
(56)

where \( \|\cdot\|_1 \) denotes the norm in \( H \). Thus by Theorem 1 there exists a continuous function \( \rho : Y \rightarrow X \) such that \( u \in H \) is a critical point \( I \) if and only if
\[ u = y + \rho(y) \] with \( y \) a critical point of \( \tilde{I}(y) = I(y + \rho(y)) \). By the continuity of the function \( b_1 \) (see Theorem 2) there exists \( \delta > 0 \) such that \( a + \delta < 1 \) and \( b + \delta < b_1(a + \delta) \). By (22), there exists a real number \( C \) such that

\[
\Gamma(t) \leq \frac{1}{2} t g_{a+\delta,b+\delta}(t) + C, \quad \text{for all } t \in \mathbb{R}. \quad (57)
\]

For \( x \in X \) and \( y \in Y \), let

\[
J_{a+\delta,b+\delta}(x + y) = \frac{1}{2} \left( B(x + y, x + y) - \int_W (x + y) g_{a+\delta,b+\delta}(x + y) \right) \quad (58)
\]

Therefore, letting \( w = r_{a+\delta,b+\delta}(y) \) we have

\[
\tilde{I}(y) = I(y + \rho(y))
\]

\[
\geq \frac{1}{2} B(y + w, y + w) - \int_W (\Gamma(y + w) + p(x, t)(y + w)) \, dx \, dt
\]

\[
\geq \frac{1}{2} \left( B(y + w, y + w) - \int_W (g_{a+\delta,b+\delta}(y + w) + p(x, t))(y + w) \, dx \, dt - 2\pi^2 C \right)
\]

\[
\geq \|y + w\|^2 \left( \frac{\tilde{J}_{a+\delta,b+\delta}(y)}{\|y + w\|^2} - \frac{\|p\|_0}{\|y + w\|_1} - \frac{2\pi^2 C}{\|y + w\|^2} \right). \quad (59)
\]

Let us see that \( \inf \{ \tilde{J}_{a+\delta,b+\delta}(y) \mid \|y\| = 1 \} \equiv A > 0 \). Let \( m = m(a + \delta) > 0 \) be as in (11). Assuming that \( \{ y_k \} \) is a sequence in \( \{ y \in Y \mid \|y\| = 1 \} \) such that \( \lim_{k \to \infty} \tilde{J}(y_k) = 0 \), by the compact imbedding of \( Y \) in \( L^2(\Omega) \) we may assume that \( \{ y_k \} \) converges weakly in \( H \) and strongly in \( L^2(\Omega) \). Let \( \tilde{\gamma} \) be such a limit and, for the sake of simplicity in the notations, let \( J_{a+\delta,b+\delta} = J, r = r_{a+\delta,b+\delta}, \) and \( \tilde{J}_{a+\delta,b+\delta} = \tilde{J} \). Arguing as in (31) we see that \( \{ r(y_k) \} \) converges in \( H \). Let \( \tilde{x} \) be such a limit. Hence, for any \( z \in X \),

\[
\langle J(\tilde{\gamma} + \tilde{x}), z \rangle_1 = B(\tilde{x}, z) - \int_W (g_{a+\delta,b+\delta}(\tilde{\gamma} + \tilde{x})) \, dz
\]

\[
= \lim_{k \to \infty} B(r(y_k), z) - \int_W (g_{a+\delta,b+\delta}(y_k + r(y_k))) \, dz
\]

\[
= 0. \quad (60)
\]
Thus $\tilde{x} = r(\tilde{y})$ and
\[
2J(\tilde{x} + \tilde{y}) = B(\tilde{x}, \tilde{x}) + B(\tilde{y}, \tilde{y}) - \int_W \left( g_{a+\delta,b+\delta}(\tilde{y} + \tilde{x}) \right) (\tilde{y} + \tilde{x}) \\
\leq \liminf_{k \to \infty} B(r(y_k), r(y_k)) + B(y_k, y_k) \\
- \int_W \left( g_{a+\delta,b+\delta}(y_k + r(y_k)) \right) (y_k + r(y_k)) \\
= \liminf_{k \to \infty} \tilde{J}(y_k) \\
= 0.
\]

Since $(a + \delta, b + \delta)$ is not in the Fucik spectrum of $\square$, we have $\tilde{x} = \tilde{y} = 0$. Thus $\lim_{k \to \infty} B(r(y_k), r(y_k)) - \int_W \left( g_{a+\delta,b+\delta}(y_k + r(y_k)) \right) (y_k + r(y_k)) = 0$. On the other hand, from the definition of $B$ (see (53)), $B(y_k, y_k) \geq \|y_k\|^2_1 = 1$, which contradicts that $\lim_{k \to \infty} \tilde{J}(y_k) = 0$. Thus $A > 0$.

Now for $y \in Y$ and $\rho(y) = w \in X$,
\[
\tilde{I}(y) = \frac{1}{2} B(y + w, y + w) - \int_W \left( \Gamma(y + w) + p(x, t)(y + w) \right) dx dt \\
\geq \frac{1}{2} \left( B(y + w, y + w) \\
- \int_W \left( g_{a+\delta,b+\delta}(y + w) + p(x, t) \right) (y + w) dx dt - 2\pi^2 C \right) \\
\geq \|y + w\|^2_1 \left( J_{a+\delta,b+\delta}(y) \frac{\|\rho\|_0}{\|y + w\|^2_1} - \frac{2\pi^2 C}{\|y + w\|^2_1} \right).
\]

From (14) we see that there exists $c > 0$, independent of $y$ such that $\|w\|_1 \leq c\|y\|_1$. These and the fact that $\tilde{J}$ is homogeneous of degree 2 (see (13)) yield
\[
\tilde{I}(y) \geq \|y + w\|^2_1 (A\|y\|^2_1 - \|p\|_0\|y + w\|_1 - 2\pi^2 C\|y + w\|^2_1) \\
\geq \|y + w\|^2_1 (A/(1 + c^2) - \|p\|_0\|y + w\|_1 - 2\pi^2 C\|y + w\|^2_1) \\
\to +\infty \quad as \quad \|y\| \to +\infty.
\]

Arguing as in Lemma 1 we see that
\[
N(y) = \frac{1}{2} B(\rho(y), \rho(y)) - \int_\Omega \left( \Gamma(\rho(y)) + pp(y) \right) d\zeta
\]
defines a weakly lower semicontinuous function. Thus $\tilde{I}$ is the sum of a convex function $(y \to B(y, y)/2 - \int_\Omega pyd\zeta)$ with a weakly lower semicontinuous function $(y \to N(y))$. Hence, by (63), $I$ achieves its minimum at some point $y_0$. By Theorem 1 we conclude that $y_0 + \rho(y_0)$ is a critical point of $I$, hence a solutions to (21). This proves Theorem 3.
Remark 3. Since $\sin(x) \in Y$, by Lemma 7, $b_1(a) < \infty$ for all $a \in (0, 1)$.

References


